International Journal of Theoretical Physics, Vol. 31, No. 2, 1992

Solutions of the Wheeler–DeWitt Equation

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Received July 16, 1991

This paper deals with the solution of the Wheeler–DeWitt equation with a massive scalar field for a Kantowski–Sachs metric using the Born–Oppenheimer approximation. Also, solutions under different assumptions are determined to indicate different stages of evolution of the universe. Finally, these wave functions in the asymptotic regions are compared with the Hartle–Hawking wave functions (including second-order corrections) evaluated using the concept of microsuperspace.

1. INTRODUCTION

An interesting and challenging problem in physics today is the applicability of quantum mechanics to the gravitational field. So among the varieties of physical problems to which quantum mechanics has been applied, those involving gravitation occupy a distinctive position. These peculiar circumstances make the usual interpretative framework of quantum mechanics vulnerable. According to DeWitt (1967), the two major difficulties are as follows:

(i) The very concept of general covariance (irrespective of the form of the action) is problematic, i.e., there is no *a priori* notion of time or position.
(ii) There is no natural probabilistic interpretation in the quantum region, as the wave equation (Wheeler–DeWitt equation) in quantum gravity is a second-order hyperbolic partial differential equation, and hence the square of the wave function cannot be taken as a probability density.

The physical consequences of quantum gravity are important when gravitational fluctuations are large. So it is very significant at very early

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0020-7748/92/0200-0289$06.50/0

stages of cosmic evolution. But if we consider the whole universe for the quantum description, further problems arise.

(iii) The idea of an external observer implicitly assumed in quantum mechanics is not applicable to the evolution of the universe, as, by definition, the universe includes everything inside it, though this may partially be resolved by the relative-state formalism of Everett (1957).

Finally, the lack of boundary conditions of the wave function of the universe creates the following difficulty:

(iv) Due to the linearity of the Hilbert space, the wave function is not unique, i.e., an infinite set of wave functions indicate the same state of the universe. So it is difficult to pick out which of these wave functions corresponds to our real universe.

Regarding the fourth problem, a very appealing proposal for the boundary condition of the universe is given by Hartle and Hawking (1983) [one may mention another interesting proposal, namely the tunneling approach by Vilenkin (1988)]. According to this proposal, the wave function for the state of minimum excitation is defined uniquely by summing over all the compact four-geometries (on the manifold M) with a given three-geometry and matter field configurations on a hypersurface (appearing in the argument of ψ_{HH}), in the Euclidean path integral formalism:

$$\psi_{\mathrm{HH}}(h_{ij}, \phi) = \int d[g_{\mu\nu}]\, d[\phi] \exp(-I_{\mathrm{E}}) \tag{1}$$

where I_{E} is the Euclidean action of the gravitational field $g_{\mu\nu}$ and the matter field ϕ (of mass m) and is given by

$$I_{\mathrm{E}} = -\frac{1}{16\pi} \int_M d^4x \sqrt{g} \cdot R - \frac{1}{8\pi} \int_S d^3x \sqrt{h} \cdot K - \int_M d^4x \sqrt{g} \cdot L_M \tag{2}$$

(the notations have their usual meaning). One can formally show that ψ_{HH} satisfies the Wheeler–DeWitt (WD) equation

$$\left\{ -G_{ijkl} \frac{\delta^2}{\delta h_{ij} \cdot \delta h_{kl}} + h^{1/2} \left[-{}^3R + \frac{1}{16\pi} T_{00}\left(\frac{\delta}{\delta\phi}, \phi \right) \right] \right\} \psi = 0 \tag{3}$$

where

$$G_{ijkl} = \tfrac{1}{2} h^{-1/2} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl})$$

and ^3R is the curvature scalar of the three-metric h_{ij} on S. The wave function is also invariant under coordinate transformations in S:

$$\left(\frac{\delta\psi}{\delta h_{ij}}\right)\bigg|_i = \tfrac{1}{2}T^{0j}$$

where a vertical stroke denotes covariant differentiation.

Now the WD equation is a functional differential equation on an infinite-dimensional space (superspace). G_{ijkl} is the metric on superspace with signature $(-+++++)$. So the WD equation can be thought of as a hyperbolic equation on superspace with $h^{1/2}$ as time coordinate. One may note that the wave function ψ is invariant under diffeomorphisms, i.e., ψ is a functional of the three-geometry and not of the particular three-metric h_{ij}. In order to solve the WD equation one must approximate it to a finite-dimensional submanifold (minisuperspace).

Moreover, the path integral (1) cannot be evaluated exactly due to indefiniteness of the measure. The qualitative behavior at sufficiently small three-geometries h_{ij} can be obtained by semiclassical approximation as (Hawking, 1984)

$$\psi_{\text{HH}} \approx P\exp(-\tilde{I}_{\text{E}}) \tag{4}$$

Here I_{E} is the Euclidean action evaluated at the compact real Euclidean solution of Einstein's equations with induced metric h_{ij} on the boundary. P is the semiclassical prefactor indicating quantum fluctuations around a classical Euclidean background. Also, for large three-geometry, the classical analogue to equation (3) is the Hamilton–Jacobi equation, which is obtained through the WKB approximation:

$$\psi = \text{Re}[C \cdot \exp(iS)] \tag{4a}$$

where C is a slowly varying function and S is the Hamilton–Jacobi function. As ψ_{HH} is real, the probability current density is identically zero; therefore no measure of probability is possible for this wave function. Thus, as far as the boundary condition is concerned, the proposal is inviting, but regarding other fundamental problems (stated above) it does not seem to offer satisfactory answers.

In Section 2 the basic equations for a Kantowski–Sachs metric are formed with some classical analysis. A detailed analysis of classical results is found in the Appendix. The wave functions based on the proposals of Hartle and Hawking (1983) and of Vilenkin (1988) are evaluated with the concept of microsuperspace (Halliwell and Louko, 1989) in Section 3. Section 4 deals with the solution of the Wheeler–DeWitt equation in the Born–Oppenheimer approximation and these results are compared with the above wave functions in the asymptotic regions. The conclusions are given in Section 5.

2. THE BASIC EQUATIONS

The four-metric in the Kantowski–Sachs model is (Chakraborty, 1990*a*)

$$dS^2 = \rho^2[-N^2(t)\, dt^2 + a^2(t)\, dr^2 + b^2(t)\, d\Omega_2^2] \tag{5}$$

Here r varies from 0 to 2π, $d\Omega_2^2$ is the metric on a unit 2-sphere, and the overall prefactor $\rho^2 = G/2\pi$ (G is the gravitational constant). The Lorentzian action and the field equations for this metric ansatz are

$$I = -\frac{1}{2}\int Nab^2\left(\frac{2}{N^2}\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{b}^2}{N^2 b^2} - \frac{1}{b^2} - \frac{\dot{\phi}^2}{N^2} + m^2\phi^2\right) dt \tag{6}$$

and

$$2\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{b}^2}{b^2} - \frac{1}{b^2} - \dot{\phi}^2 - m^2\phi^2 = 0 \tag{7}$$

$$2\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{1}{b^2} + \dot{\phi}^2 - m^2\phi^2 = 0 \tag{8}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} + \dot{\phi}^2 - m^2\phi^2 = 0 \tag{9}$$

$$\ddot{\phi} + \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)\dot{\phi} + m^2\phi = 0 \tag{10}$$

($\cdot \equiv d/dt$). Equation (7) is the scalar constraint equation, and the equation of continuity is given by equation (10). The Wheeler–DeWitt equation is obtained from the constraint equation, replacing the momentum variables by the corresponding quantum mechanical operator, and the form is

$$\left[-\left(\frac{a}{2b^2}\right)\frac{1}{a^p}\frac{\partial}{\partial a}\left(a^p\frac{\partial}{\partial a}\right) + \frac{1}{b}\frac{1}{b^q}\frac{\partial}{\partial b}\left(b^q\frac{\partial}{\partial a}\right) - \frac{1}{2ab^2}\frac{\partial^2}{\partial\phi^2} - \frac{a}{2} + \frac{1}{2}m^2\phi^2 ab^2\right]\psi = 0 \tag{11}$$

Here p, q denote some of the ambiguities in the factor ordering. The wave function ψ is a function of three variables: $\psi \equiv \psi(a, b, \phi)$. Hence in this model the superspace is reduced to a three-dimensional manifold, $0 \le a$, $b < \infty$, $-\infty < \phi < +\infty$.

Since the values of p, q do not affect the nature of the solution to a great extent, we take p to be 1 and q to be 1/2. Hence the above equation becomes

$$\left[2ab\frac{\partial^2}{\partial a\,\partial b}-a^2\frac{\partial^2}{\partial a^2}-\frac{\partial^2}{\partial\phi^2}-a^2b^2+m^2\phi^2a^2b^4\right]\psi=0 \tag{12}$$

Let us now substitute

$$a=e^{\alpha},\qquad b=e^{\beta-\alpha}$$

Then the d'Alambertian becomes diagonal and the resulting WD equation is

$$\frac{\partial^2\psi}{\partial\beta^2}-\frac{\partial^2\psi}{\partial\alpha^2}-\frac{\partial^2\psi}{\partial\phi^2}-e^{2\beta}\psi+m^2\phi^2\,e^{2(2\beta-\alpha)}\psi=0 \tag{13}$$

The initial conditions for the Lorentzian trajectories which are solutions of the field equations are (Chakraborty, 1990*a*)

$$\begin{array}{lll} a(0)=0, & b(0)=1/\delta, & \phi(0)=\phi_0 \\ \dot{a}(0)=\mu\delta, & \dot{b}(0)=0, & \dot{\phi}(0)=0 \end{array} \tag{14}$$

(here the argument 0 stands for $t=0$), where μ is arbitrary and $\delta=m\phi_0/\sqrt{3}$. The Hamilton–Jacobi function, which is the action corresponding to Lorentzian trajectories, is given by (Chakraborty, 1990*a*)

$$S=-ab(\tfrac{1}{3}m^2\phi^2b^2-1)^{1/2} \tag{15}$$

Hence in the WKB approximation the wave function in the classical region is [see equation (4a)]

$$\psi\simeq\mathrm{Re}[C\exp(iS)] \tag{16}$$

3. WAVE FUNCTIONS IN KANTOWSKI–SACHS MICROSUPERSPACE MODEL

This section is a review of the work of Chakraborty (1990*b*). The Euclidean four-metric in the Kantowski–Sachs ansatz is

$$dS^2=\rho^2[N^2(\tau)\,d\tau^2+a^2(\tau)\,dr^2+b^2(\tau)\,d\Omega_2^2] \tag{17}$$

The Einstein–Hilbert section for the metric ansatz (17) in the Euclidean version is

$$I_E = \frac{1}{2}\int_0^{T^*} N\,d\tau\left(-a - 2\dot{a}\frac{\dot{b}b}{N^2} - \frac{a\dot{b}^2}{N^2} - H^2ab^2\right) \tag{18}$$

($\cdot \equiv d/d\tau$ in this section). Here, $H^2 = m^2\phi^2$ is constant, as the scalar field ϕ is very large and almost constant at very early stages of the evolution (Chakraborty, 1990*a*). Accordingly, the Euclidean field equations are

$$2b\ddot{b} + \dot{b}^2 = N^2(1 - H^2b^2) \tag{19}$$

$$\ddot{a}b + b\ddot{a} + \dot{a}\dot{b} + N^2H^2ab = 0 \tag{20}$$

and

$$a - \frac{a\dot{b}^2}{N^2} - \frac{2\dot{a}\dot{b}b}{N^2} - H^2ab^2 = 0 \tag{21}$$

The solution for these field equations are

$$a(\tau) = a_1 \cos\left(\frac{N\tau H}{\sqrt{3}}\right) \tag{22}$$

$$b(\tau) = \frac{\sqrt{3}}{H}\sin\left(\frac{N\tau H}{\sqrt{3}}\right) \tag{23}$$

where the boundary conditions based on the HH proposal are

$$a = a_1, \qquad \dot{a} = 0, \qquad b = 0, \qquad \dot{b} = N \qquad \text{at} \quad \tau = 0 \tag{24}$$

(a_1 is arbitrary).

In the concept of microsuperspace, the four-metrics are labeled by an arbitrary parameter (say b_1). So the class of four-metrics under consideration have the following scale factors:

$$a(\tau) = a_1 \cos\left(\frac{N\pi}{b_1}\right) \tag{25}$$

$$b(\tau) = b_1 \sin\left(\frac{N\tau}{b_1}\right) \tag{26}$$

Thus the action for this class of metric is

$$I_E(b_1) = \frac{a_1b_1}{2}\left[-2\sin\left(\frac{NT^*}{b_1}\right) + \frac{\sin^3(NT^*/b_1)}{3}(H^2b_1^2 + 3)\right] \tag{27}$$

We now define

$$c = b(T^*) = b_1 \sin(NT^*/b_1) \tag{28}$$

$$d = a(T^*) = a_1 \cos(NT^*/b_1) \tag{29}$$

and a new variable

$$Z = \cos(NT^*/b_1) \tag{30}$$

So the action (27) now simplifies to

$$I_E(z, c, d) = \frac{cd}{6}\left(\frac{H^2C^2 - 3}{z} - 3z\right) \tag{31}$$

Hence the path integral expression of the wave function based on the Hartle and Hawking (1983) proposal is now reduced to a single ordinary integration over z as

$$\psi(c, d) = \int_\Gamma dz\, \mu(z, c, d) \exp[-I_E(z, c, d)] \tag{32}$$

where μ is a measure of integration and Γ lies in the complex z plane such that (32) converges. Now the integration over z is evaluated by the method of steepest descent; for detailed analysis see Chakraborty (1990*b*). The classically forbidden and allowed regions correspond to $H^2C^2 < 3$ and $H^2C^2 \gtrless 3$, respectively.

The wave functions (to second order) in these regions are

$$[1/(cd)^{1/2}](1 - H^2C^2/3)^{1/4} \exp(-I_\pm) \tag{33}$$

and

$$[1/(cd)^{1/2}](H^2C^2/3 - 1)^{1/4} \exp(-I^c_\pm) \tag{34}$$

or

$$\frac{1}{(cd)^{1/2}}\left(\frac{H^2C^2}{3} - 1\right)^{1/4} \cos\left[cd\left(\frac{H^2C^2}{3} - 1\right)^{1/2} + \frac{\pi}{4}\right] \tag{35}$$

respectively, with

$$I_\pm = \mp cd\left(1 - \frac{H^2C^2}{3}\right)^{1/2}$$

$$I^c_\pm = \mp icd\left(\frac{H^2C^2}{3} - 1\right)^{1/2}$$

for different choices of the convergent contours. So the wave function based on the HH proposal is not unique; it depends on the choice of the contour (Halliwell and Louko, 1989).

The wave function due to the approach of Vilenkin (1988) does not give any boundary condition, but determines the unique contour and is given by

$$\frac{1}{(cd)^{1/2}}\left(1-\frac{H^2C^2}{3}\right)^{1/4}\exp(-I_-) \qquad \text{for} \quad H^2C^2<3 \tag{36}$$

and

$$\frac{1}{(cd)^{1/2}}\left(\frac{H^2C^2}{3}-1\right)^{1/4}\exp(-iI_-^c) \qquad \text{for} \quad H^2C^2>3 \tag{37}$$

Finally, we note that the measure μ is taken to be one throughout the calculation. Different choices of μ only change the phase factor of the wave function.

4. SOLUTION OF THE WHEELER-DEWITT EQUATION

In this section we solve the Wheeler-DeWitt (WD) equation (13) using the Born-Oppenheimer approximation (Kiefer, 1988). The motivation for performing the Born-Oppenheimer approximation in the present case is that gravitational degrees of freedom can be regarded as heavier than matter degrees of freedom. Accordingly, let

$$\psi(\alpha, \beta, \phi)=\sum_n C_n(\alpha, \beta)\Phi_n(\alpha, \beta, \phi) \tag{38}$$

where the Φ_n depend adiabatically on α, β and are the eigenfunctions of the Hamiltonian for the damped harmonic oscillator:

$$H_{\text{red}}=-\frac{\partial^2}{\partial\phi^2}+\omega^2(\alpha, \beta)\phi^2-e^{2\beta} \tag{39}$$

This harmonic oscillator in ϕ has the frequency

$$\omega(\alpha, \beta)=m\, e^{2\beta-\alpha}$$

and the eigenvalue equation is

$$H_{\text{red}}\Phi_n(\alpha, \beta, \phi)=E_n(\alpha, \beta)\Phi_n(\alpha, \beta, \phi)$$

The energy eigenvalues $E_n(\alpha, \beta)$ also depend adiabatically on α, β and are given by

$$E_n(\alpha, \beta)=(2n+1)m\, e^{2\beta-\alpha}-e^{2\beta} \tag{40}$$

The energy eigenfunctions $\Phi_n(\alpha, \beta, \phi)$ have the expression

$$\Phi_n(\alpha, \beta, \phi) = \left(\frac{\omega(\alpha, \beta)}{\pi}\right)^{1/4} \frac{1}{(2^n n!)^{1/2}} H_n(\phi[\omega(\alpha, \beta)]^{1/2}) \times \exp\left[-\omega(\alpha, \beta)\frac{\phi^2}{2}\right] \tag{41}$$

(the notations have their usual meaning). Inserting (38) into (13) yields

$$\sum_n \Phi_n \left[\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2} + E_n(\alpha, \beta)\right] C_n(\alpha, \beta) + 2\sum_n \left(\frac{\partial C_n}{\partial\beta}\frac{\partial\Phi_n}{\partial\beta} - \frac{\partial C_n}{\partial\alpha}\frac{\partial\Phi_n}{\partial\alpha}\right) + \sum_n C_n \left(\frac{\partial^2\Phi_n}{\partial\beta^2} - \frac{\partial^2\Phi}{\partial\alpha^2}\right) = 0$$

Taking the scalar product with Φ_e and using the orthonormal property of the eigenfunctions $\{\Phi_n\}$, we have

$$\left\{\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2} + E_e(\alpha, \beta)\right\} C_e(\alpha, \beta) + 2\sum_n \int \Phi_e^* \left(\frac{\partial C_n}{\partial\beta}\frac{\partial\Phi_n}{\partial\beta} - \frac{\partial C_n}{\partial\alpha}\frac{\partial\Phi_n}{\partial\alpha}\right) d\phi + \sum_n \int \Phi_e^* C_n \left(\frac{\partial^2\Phi_n}{\partial\beta^2} - \frac{\partial^2\Phi_n}{\partial\alpha^2}\right) d\phi = 0$$

Neglecting the coupling terms, according to the Born–Oppenheimer approximation, the above equation simplifies to the two-dimensional partial differential equation

$$\left\{\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2} + E_n(\alpha, \beta)\right\} C_n(\alpha, \beta) = 0 \tag{42}$$

This is similar to the WD equation for the FRW metric.

We now make a similar expansion for $C_n(\alpha, \beta)$ with α as the adiabatic parameter:

$$C_n(\alpha, \beta) = \sum_k \xi_k(\alpha)\eta_k(\alpha, \beta) \tag{43}$$

where $\eta_k(\alpha, \beta)$ are the eigenfunctions of the differential equation

$$\frac{\partial^2\eta_k}{\partial\beta^2} + e^{2\beta}\theta(\alpha)\eta_k = \lambda_k^2\eta_k \qquad (\lambda_k \text{ is arbitrary})$$

with

$$\theta(\alpha) = (2n+1)m\, e^{-\alpha} - 1$$

So by the substitution

$$Z = e^{\beta}$$

the eigenvalue equation reduces to a Bessel (or modified Bessel) equation and we have

$$\begin{aligned} \eta_k(\alpha, \beta) &= J_{\lambda_k}(e^{\beta}\sqrt{\theta}), \qquad \theta > 0 \\ &= I_{\lambda_k}(e^{\beta}\sqrt{|\theta|}), \qquad \theta < 0 \end{aligned} \tag{44}$$

Again substituting (43) in (42) and using the above eigenvalue equation, we have

$$-\sum_k \eta_k(\alpha, \beta)\left(\frac{d^2\xi_k}{d\alpha^2} - \lambda_k^2 \xi_k\right) - 2\sum_k \frac{\partial \xi_k}{\partial \alpha}\frac{\partial \eta_k}{\partial \alpha} - \sum_k \xi_k \frac{\partial^2 \eta_k}{\partial \alpha^2} = 0$$

Taking the scalar product with $\eta_l(\alpha, \beta)$ (with the orthogonal property) and neglecting the cross terms, we find the differential equation for ξ_k:

$$\frac{d^2\xi_k}{d\alpha^2} - \lambda_k \xi_k = 0$$

i.e., $\xi_k = \exp(\pm\lambda_k \alpha)$.

Hence the general solution of the Wheeler–DeWitt equation (13) is

$$\begin{aligned} \psi(\alpha, \beta, \phi) = \sum_n \sum_k \left[\frac{\omega(\alpha, \beta)}{\pi}\right]^{1/4} \frac{1}{(2^n n!)^{1/2}} H_n(\phi[\omega(\alpha, \beta)]^{1/2}) \\ \times \exp\{-[\omega(\alpha, \beta)/2]\phi^2\}\{J_{\lambda_k}(e^{\beta}\sqrt{\theta}) \text{ or } I_{\lambda_k}(e^{\beta}\sqrt{|\theta|})\} \exp(\pm\lambda_k \alpha) \end{aligned}$$

where n takes positive integral values or zero, but k may take for convenience integral values (including zero).

The oscillatory nature of J_n (or exponential nature of I_γ) in the asymptotic region shows that the above solutions of the Wheeler–DeWitt equation behave correctly in the classical (or classically forbidden) region. Thus, the nature of the solutions of the Wheeler–DeWitt equation (by the Born–Oppenheimer approximation) is identical to that of those derived from the path integral formalism and also with equation (4a) in the classical solution.

5. CONCLUSION

The solution of the Wheeler–DeWitt equation, evaluated using the Born–Oppenheimer approximation, agrees qualitatively with the wave function in the path integral formalism. An exact comparison is not possible due to the arbitrary measure in the path integral and the factor ordering in the Wheeler–DeWitt equation. Moreover, the path integral and the Wheeler–DeWitt equation are evaluated approximately.

In the Born–Oppenheimer approximation the treatment of α as the adiabatic parameter instead of β in equation (43) has no physical reason; it is only for reasons of mathematical simplicity. Also, the solution of the Wheeler–DeWitt equation using the above approximation is complicated and lengthy in form. So it is very difficult to draw any definite conclusion from it. Therefore, for future work it would be interesting to calculate the solution of the WD equation in compact form, so that some definite relation can be drawn between the path integral measure and the factor ordering index, comparing with the wave function in the path integral formulation.

APPENDIX. A STUDY OF CLASSICAL FIELD EQUATIONS

The classical field equations (8)–(10) reduce to the first-order coupled equations

$$\frac{dx}{d\eta}=y \tag{A1}$$

$$\frac{dy}{d\eta}=-x-y(z+2u) \tag{A2}$$

$$\frac{dz}{d\eta}=-z^2+x^2-2zu \tag{A3}$$

$$\frac{du}{d\eta}=-u^2+zu-y^2 \tag{A4}$$

by the substitution

$$\begin{gathered} x\equiv\phi, \qquad y\equiv m^{-1}\dot{\phi}, \qquad z\equiv m^{-1}\dot{\alpha}, \qquad u\equiv m^{-1}\dot{\beta} \\ \eta\equiv mt, \qquad \alpha=\ln a, \qquad \beta=\ln b \end{gathered} \tag{A5}$$

The solutions of this set of first-order equations contain three arbitrary constants and represent three-parameter congruence of trajectories in the (x, y, z, u) space, as they satisfy the constraint equation

$$\frac{1}{m^2 e^{2\beta}}=x^2+y^2+z^2-(z+u)^2 \tag{A6}$$

The metric on superspace is

$$dS^2=(e^{2(\alpha+2\beta)}m^2\phi^2-e^{2(\alpha+\beta)})(d\alpha^2+d\phi^2-\tfrac{1}{2}d\alpha\, d\beta) \tag{A7}$$

If we introduce

$$\gamma=\alpha-\beta/4, \qquad \delta=\beta/4$$

then the auxiliary metric becomes

$$dS^2=[e^{2(\gamma+9\delta)}m^2\phi^2-e^{2(\gamma+5\delta)}](d\gamma^2+d\phi^2-d\delta^2) \tag{A8}$$

So the geodesic equations for this auxiliary metric can be written (after eliminating the affine parameter) as two second-order equations (Page, 1984):

$$\frac{d^2\gamma}{d\delta^2}=-\frac{1-(d\gamma/d\delta)^2-(d\phi/d\delta)^2}{e}\left(e+f\frac{d\gamma}{d\delta}\right) \tag{A9}$$

and

$$\frac{d^2\phi}{d\delta^2}=-\frac{1-(d\gamma/d\delta)^2-(d\phi/d\delta)^2}{e}\left(g+f\frac{d\phi}{d\delta}\right) \tag{A10}$$

with

$$e=e^{2(\gamma+5\delta)}-m^2\phi^2\, e^{2(\gamma+9\delta)}$$

$$f=5e^{2(\gamma+5\delta)}-9m^2\phi^2\, e^{2(\gamma+9\delta)}$$

$$g=-m^2\phi\, e^{2(\gamma+9\delta)}$$

The constrained phase space can be classified from the point of view of temporal oscillations and the stress-energy of the homogeneous scalar field ϕ. The stress-energy tensor has the form of a perfect fluid at rest with energy density and pressure (Hawking and Page, 1988)

$$\rho=\tfrac{1}{2}(\dot\phi^2+m^2\phi^2)=\tfrac{1}{2}m^2(x^2+y^2) \tag{A11}$$

$$p=\tfrac{1}{2}(\dot\phi^2-m^2\phi^2)=\tfrac{1}{2}m^2(y^2-x^2) \tag{A12}$$

The field equation for ϕ [equation (10)] shows that the oscillations of ϕ are overdamped or underdamped accordingly as

$$\dot\alpha^2+2\dot\beta^2\gtrless\tfrac{4}{9}m^2$$

In the stiff region (i.e., when ϕ is strongly overdamped and $p\simeq\rho$) terms containing m^2 are insignificant. So, if we neglect the mass term in the action (6), then

$$p_\phi=ab^2\dot\phi=\text{const} \tag{A13}$$

Moreover, setting $m^2=0$, the auxiliary metric (A8) becomes flat and the equations for geodesics simplify to

$$\ddot{\gamma}=-(1-\dot{\gamma}^2-\dot{\phi}^2)(1+5\dot{\gamma})$$
$$\ddot{\phi}=-(1-\dot{\gamma}^2-\dot{\phi}^2)\cdot 5\dot{\phi}$$

$(\cdot=d/d\delta)$. So the trajectories are planes in the three-dimensional space (γ, ϕ, δ) of the form

$$A\phi+5\gamma+\delta=B \tag{A14}$$

The two arbitrary constants A and B parametrize the solution.

Because the auxiliary metric (A8) is conformally flat, the trajectories may also be interpreted as those of a particle of variable mass squared,

$$M^2=e^{2(\gamma+9\delta)}m^2\phi^2-e^{2(\gamma+5\delta)} \tag{A15}$$

moving in the flat three-dimensional Minkowski metrix $-d\delta^2+d\gamma^2+d\phi^2$. At $b^2=1/m^2\phi^2$, M^2 vanishes and the auxiliary metric (A8) and the geodesics (A9) and (A10) are singular. However, the trajectories simply pass through these curves and change from timelike ($|bm\phi|<1$) to spacelike ($|bm\phi|>1$). The physical metric (2.1) is, however, regular here, but it has a singularity at a or $b=0$.

ACKNOWLEDGMENTS

The author is grateful to the Relativity–Cosmology Centre in the Department of Physics, Jadavpur University, for helpful discussions. He also thanks the UGC–DSA Programme in the Department of Mathematics for financial assistance.

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